Brownian motion and Stochastic Calculus
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Assignment 8

Exercise 1

Let a and d be positive real numbers and B a standard Brownian motion.

1) Compute for $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_d|\sqrt{2\lambda}\big)\mathbf{1}_{\{B_d \leq -a\}}\Big].$$

2) Define $T_1 := \inf\{t \geq d : B_t = 0\}$. Show that T_1 is an $\mathbb{F}^{B,\mathbb{P}}$–stopping time, and compute for any $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\big], \text{ and } \mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\mathbf{1}_{\{B_d \leq -a\}}\big].$$
Show then that B_{T_1+d} is independent of B_d and T_1.

3) We now define τ_1 by
$$\tau_1 := \begin{cases} d, \text{ if } B_d \leq -a, \\ T_1 + d, \text{ if } B_d > -a, \text{ and } B_{T_1+d} \leq -a, \\ +\infty, \text{ otherwise.} \end{cases}$$
Compute for any $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda \tau_1)\big].$$

4) Let now
$$T_2 := \inf\{t \geq T_1 + d : B_t = 0\}.$$
As above we then introduce
$$\tau_2 := \begin{cases} d, \text{ if } B_d \leq -a, \\ T_1 + d, \text{ if } B_d > -a, \text{ and } B_{T_1+d} \leq -a, \\ T_2 + d, \text{ if } B_d > -a, \ B_{T_1+d} > -a, \text{ and } B_{T_2+d} \leq -a, \\ +\infty, \text{ otherwise.} \end{cases}$$
Show that B_{T_2+d} is independent of (B_{T_1+d}, B_d) and T_2, then compute for any $\lambda > 0$
$$\mathbb{E}\big[\exp(-\lambda \tau_2)\big].$$

Exercise 2

Let B be a standard one-dimensional Brownian motion $(B_t)_{t \geq 0}$. We define
$$X_t := \frac{1}{t}\int_0^t \mathbf{1}_{\{B_s > 0\}} \mathrm{d}s, \ t > 0.$$
Our goal is to show that
$$\mathbb{P}[X_t < u] = \frac{2}{\pi}\mathrm{Arcsin}(\sqrt{u}), \ 0 \leq u \leq 1, \ t > 0.$$

1) What does X_t represent?

2) Show that the law of X_t is equal to the law of X_1, for any $t > 0$.

3) We fix $\lambda > 0$ and define for $(t,x) \in]0,+\infty[\times\mathbb{R}$ the map

$$v(t,x) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\lambda\int_0^t \mathbf{1}_{\{x+B_s>0\}}\mathrm{d}s\right)\right],$$

as well as its Laplace transform

$$g_\rho(x) := \int_0^{+\infty} v(t,x)\mathrm{e}^{-\rho t}\mathrm{d}t,\ \rho > 0.$$

Show that

$$g_\rho(0) = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\rho + \lambda X_1}\right].$$

4) Assuming that all functions appearing are smooth enough, show that v must satisfy

$$\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) - \lambda\mathbf{1}_{\{x>0\}}v(t,x).$$

5) Deduce then that g_ρ must satisfy

$$g_\rho''(x) = -2 + 2\rho g_\rho(x) + 2\lambda\mathbf{1}_{x>0}g_\rho(x).$$

6) Solve this ODE on $\mathbb{R}$, and deduce in particular that

$$g_\rho(0) = \frac{1}{\sqrt{\rho(\lambda+\rho)}}.$$

7) Deduce that the result stated at the beginning of the exercise holds. You may want to use (and prove!) the following identity

$$\frac{1}{\sqrt{1+\lambda}} = \frac{1}{\pi}\sum_{n=0}^{+\infty}(-\lambda)^n\int_0^1 \frac{x^n}{\sqrt{x(1-x)}}\mathrm{d}x.$$